Sandpile groups of generalized de Bruijn and Kautz graphs and circulant matrices over finite fields

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June 3, 2014

What is Sandpile?
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Sandpile Group

Definition (Laplacian Matrix)

Let $G = (V, E)$ be a directed graph. The **Laplacian matrix** $\Delta(G)$ of size $|V| \times |V|$ is defined as follows:

$$\Delta(G)_{i,j} := \begin{cases} 
- \text{# of edges from } v_j \text{ to } v_i & \text{if } i \neq j \\
\text{outdeg}(i) - \text{# of loop at vertex } v_i & \text{if } i = j.
\end{cases}$$
Sandpile Group

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Example

$$\Delta(G) = \begin{bmatrix}
2 & -1 & -1 & 0 \\
-1 & 2 & -1 & 0 \\
-1 & 0 & 2 & -1 \\
0 & -1 & 0 & 1
\end{bmatrix}$$
Sandpile Group (continued)

Definition (Sandpile Group)

Let $G$ be an Eulerian digraph ($\text{indeg}(v) = \text{outdeg}(v)$). Fix a vertex $v$, define $L'(G, v)$ to be the subgroup of $\mathbb{Z}^{|V(G)|}$ generated by the columns of the principal minor $\Delta(G)$ formed by deleting the row and columns corresponding to $v$.

**Sandpile group (or a critical group)** $S(G)$ of $G$ is defined as the quotient group

$$S(G) := \mathbb{Z}^{|V(G)|}/L'(G, v).$$
Definition (Sandpile Group)

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**Sandpile group (or a critical group) $S(G)$ of $G$ is defined as the quotient group**

$$S(G) := \mathbb{Z}^{|V(G)|-1}/L'(G, v).$$

Remark

The sandpile group $S(G)$ does not depend on the choice of $v$ for $L'(G, v)$ if $G$ is an Eulerian graph.
Sandpile Group (continued)

\[
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2 & -1 & -1 & 0 \\
-1 & 2 & -1 & 0 \\
-1 & 0 & 2 & -1 \\
0 & -1 & 0 & 1
\end{bmatrix}
\]

\[
S(G) = \mathbb{Z}^3/L'(G, 1) = \mathbb{Z}^3/\left\langle \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\rangle = \mathbb{Z}_3
\]

\[
S(G) = \mathbb{Z}^3/L'(G, 2) = \mathbb{Z}^3/\left\langle \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\rangle = \mathbb{Z}_3
\]
Applications of Sandpile Groups

Several applications of Sandpile Group:

1. algebraic model of chip-firing games;
2. load-balancing in multiprocessor systems (Computing);
3. test-bed for the concept of self-organized criticality (Physics/CS).
Graph DB\((n, d)\)

Definition (Graph DB\((n, d)\))

Let \(d\) be a fixed natural number. **Generalized de Bruijn Graph** DB\((n, d)\) is a directed graph with \(V = \mathbb{Z}_n\) as the vertex set, and there is an arc from \(v_1\) to \(v_2\) if and only if

\[
v_2 - dv_1 \equiv 0, 1, \ldots, d - 1 \mod n.
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\]

Definition (Graph Ktz\((n, d)\))

**Generalized Kautz Graph** Ktz\((n, d)\) is a directed graph with \(V = \mathbb{Z}_n\) as the vertex set, and there is an arc from \(v_1\) to \(v_2\) if and only if

\[
v_2 + dv_1 \equiv 0, 1, \ldots, d - 1 \mod n.
\]
DB(6, 2) (i.e. $n = 6$ and $d = 2$).

Example

![Diagram of a network with nodes labeled 0, 1, 2, 3, 4, and 5 connected by edges.](image_url)
For $d$ prime DB($d^k$, $d$) and Ktz($d^k$, $d$) are known as the “classical” de Bruijn graph and the Kautz graph respectively.
In this case the sandpile group was explicitly computed by L. Levine, H. Bidhori and S. Kishore.
One of our main results is generalizing their results to the case of arbitrary value of $d$ and $n$. 
Application of $\text{DB}(n, d)$ and $\text{Ktz}(n, d)$

Some areas in "real" life in which de Bruijn Graphs and Kautz Graphs are useful:

1. Parallel computing (Grid network topologies);
2. The distributed hash table protocol (e.g. Koorde);
3. De novo assembly of (short) read sequences into a genome (Bioinformatics).
First Main Result

Theorem 1

The group $S(DB(n, d)) = S(n, d)$ is given by

$$S(n, d) = \left( \bigoplus_{i=0}^{k-1} \mathbb{Z}_{d^{i+1}} / g_i \right) \oplus \mathbb{Z}_{d^{i+1}}^{n-2n+2n+2-1} \bigoplus_{\nu \in V} \mathbb{Z}_{(d^o(\nu) - 1)/c(\nu)}.$$

Remark

$S(Ktz(n, d))$ has the same decomposition as above, with $d$ substituted with $-d$ in defining $V$ and $c(\nu)$. 
Introduction

Results

Further Research

Notation used in Theorem 1

Definition \((n_i \text{ and } g_i)\)

\[\{n_i \mid i \geq 0\} \text{ and } \{g_i \mid i \geq 0\} \text{ are defined recursively:}\]

\[n_0 = n; \quad n_{i+1} = \frac{n_i}{g_i}; \quad g_i = \gcd(n_i, d).\]

Definition \((O(v) \text{ and } V)\)

Let \(k \geq 0\) be the smallest integer for which \(g_k = 1\). The map \(x \mapsto dx\) partitions \(\mathbb{Z}_{n_k}\) into orbits of the form

\[O(v) = \{v, dv, \ldots, d^{o(v)-1}v\}.\] (Sometimes called \(d\)-ary cyclotomic coset of \(v\).)

Set \(o(v) := |O(v)|\), pick a transversal \(V \subset \mathbb{Z}_{n_k}\) of the orbits \(O(v)\).
Notation used in Theorem 1 (continued)

**Definition (c(v))**

For a prime $p$, let $\pi_p(m) := \max \ell \ p^\ell | m$. For any $p | m$, define $c(v)$ as:

$$c(v) = \begin{cases} 
\pi_p(m) & v = m/\pi_p(m), \ p \neq 2 \text{ or } d \equiv 1 \mod 4 \text{ or } 4 \nmid m \\
\pi_2(m)/2 & v = m/\pi_2(m), \ d \equiv 3 \mod 4, \text{ and } 4 \mid m \\
2 & v = m/2, \ 4 \mid m \text{ and } d \equiv 3 \mod 4 \\
1 & \text{otherwise.}
\end{cases}$$
Question:
Why do we want to compute the sandpile group of generalized de Bruijn Graphs?
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Answer:
For $d = p^k$, it appears to be connected to another interesting object: invertible circulant matrices, a.k.a. units in the group algbera of $\mathbb{Z}_n$ over $\mathbb{F}_{p^k}$. 
**Circulant Matrices**

**Definition (Circulant Matrix)**

A $n \times n$ matrix $A$ is called **circulant** if it is of the following form:

$$
A = \begin{bmatrix}
    a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\
    a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\
    \vdots & \ddots & \ddots & \ddots & \ddots \\
    a_2 & \cdots & \cdots & a_1 \\
    a_1 & a_2 & \cdots & a_{n-1} & a_0
\end{bmatrix}.
$$
Definition \((Q_n)\)

We define \(Q_n := (q_{i,j})_{0 \leq i,j < n}\) to be the \(n \times n\) matrix that satisfies

\[
q_{i,j} = \begin{cases} 
1 & \text{if } j \equiv i + 1 \mod n, \\
0 & \text{otherwise}.
\end{cases}
\]
Circulant Matrices (continued)

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0 & \text{otherwise}.
\end{cases}$$

**Definition ($C(n, p)$ and $C'(n, p)$)**

We define the group $C(n, p)$ as the group of $n \times n$ invertible circulant matrices over $\mathbb{F}_p$. Let $\mathbf{1} = (1, \ldots, 1)^\top$. Define

$$C'(n, p) := \{ g \in C(n, p) \mid g \mathbf{1} = \mathbf{1} \}.$$
Circulant Matrices (continued)

Some interesting facts about $C(n, p)$:

1. is commutative
2. acts regularly on the normal polynomials of degree $n$ over $\mathbb{F}_p$
3. Connected with the cyclic codes over $\mathbb{F}_p$
Second Main Result

Theorem 2

Let $d$ be prime. Then

$$S(DB(n, d)) \cong C'(n, d)/\langle Q_n \rangle.$$
Second Main Result

Theorem 2

Let $d$ be prime. Then

$$S(DB(n, d)) \simeq C'(n, d)/\langle Q_n \rangle.$$ 

Remark

Can one provide a bijective proof of this?!
Further research and possible applications

Possible continuation of this research:

- Find (???) a new algorithm to construct normal polynomials.
- Analysis on the sandpile group of other family of graphs similar to deBruijn Graphs.
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Epilogue

For further enquiries:
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THANK YOU!