Spectral Properties of Simplicial Rook Graphs

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Tilburg University, Netherlands

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### Definition (Simplicial Rook Graph)

The *simplicial rook graph* $SR(d, n)$ is the graph whose vertices are lattice points in the $n$th dilate of the standard simplex in $\mathbb{R}^d$, with two vertices adjacent if and only if they differ by a multiple of $e_i - e_j$ for some pair $i, j$. 
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![Diagram of simplicial rook graph]
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SR(2, n) \cong K_{n+1}, \text{ since } V(2, n) = \{(x, y) \mid x, y \geq 0, x + y = n\}.
SR\((d, n)\) for small \(d\) or \(n\)

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- \(SR(d, 1) \cong K_d\), since \(V(d, 1) = \{e_1, \ldots, e_d\}\).
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- $SR(d, 2) \cong J(d + 1, 2) \cong T(d + 1)$. Why?
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Martin and Wagner’s Results

- $SR(d, n)$ has $\binom{n+d-1}{d-1}$ vertices.
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When $n < \binom{d}{2}$, the smallest eigenvalue in all known cases is $-\binom{d}{2}$ with multiplicity the Mahonian number $M(d, n)$. 
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The spectrum of $SR(3, n)$ is:

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If $n = 2m + 1$: If $n = 2m$: When $d = 4$, the spectrum is integral for $n \leq 30$. When $d = 5$, the spectrum is integral for $n \leq 25$.
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When $d = 4$, the spectrum is integral for $n \leq 30$.

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Martin and Wagner conjecture that the spectrum of $SR(d, n)$ is always integral.
For fixed $d, n$, we partition $V(d, n)$ into subsets $V_1, V_2, \ldots$ where $V_i$ is the set of all vertices with exactly $i$ nonzero coordinates.

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The quotient matrix of this partition is

\[
Q = \begin{pmatrix}
  a_1 & b_1 & 0 & \cdots & 0 \\
  c_2 & a_2 & b_2 & \ddots & \vdots \\
  0 & c_3 & \ddots & \ddots & 0 \\
  \vdots & \vdots & \ddots & a_{m-1} & b_{m-1} \\
  0 & \cdots & 0 & c_m & a_m
\end{pmatrix},
\]

where $a_i = (n - i)(i - 1) + i(d - i)$, $b_i = (n - i)(d - i)$, $c_i = i(i - 1)$, and $m = \min\{n, d\}$. 
Partial Spectrum of $SR(d, n)$

- Every eigenvalue of a quotient matrix of an equitable partition of a graph is also an eigenvalue of the adjacency matrix, so:

$$\mu_i = (d - i)n - (i - 1)(d - (i - 1))$$ is an eigenvalue of $SR(d, n)$.

The proof includes the eigenvectors of $Q$, which can be extended to eigenvectors of $SR(d, n)$. 

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Simplicial Rook Graphs
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**Proposition**

For fixed \( n, d \), let \( m = \min\{n, d\} \). For each \( i \in [m] \),

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**Partial Spectrum of $SR(d, n)$**
Diameter of $SR(d, n)$

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Key facts for the proof:

- The diameter is trivially at most $n$, and (if $n < d$) the vertices $(n, 0, \ldots, 0)$ and $(0, 1, \ldots, 1, 0, \ldots, 0)$ are at distance $n$. 
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- A vertex in $V_i$ only has neighbors in $V_{i-1}$, $V_i$, and $V_{i+1}$, so the diameter is at least $d - 1$ if $n \geq d$. 
Proposition

For any fixed $n, d$, the clique number of $SR(d, n)$ is $\max\{d, n + 1\}$.

- The set $V_1$ is a clique of size $d$, while the set
  \[ \{(x, y, 0, \ldots, 0) \mid x, y \geq 0, x + y = n \} \]
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![Graph of SR(d, n)](image)
Martin and Wagner asked for what values of $d$ and $n$ the graph $SR(d, n)$ is DS.
When is $SR(d, n)$ Determined by its Spectrum (DS)?

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- $SR(d, 2) \cong T(d + 1)$ is DS unless $d = 7$ since the triangular graph $T(k)$ is DS unless $k = 8$. 
Using Godsil-McKay switching we find that:

- $SR(4, n)$ is not DS for $n \geq 3$.
  - $V_1$ is a Godsil-McKay switching set.
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We have found 3 nonisomorphic graphs with the spectrum of $SR(4, 3)$. 
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- Found a common equitable partition.
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To do this, we:

- Found a correspondence between vertices in $SR(d, 3)$ and $J(d + 2, 3)$.
- Found a common equitable partition.
- Built spectrum of $SR(d, 3)$ from that of $J(d + 2, 3)$. 
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These subgraphs are bipartite, $n$-regular, and give $SR(d, n)$ the eigenvalue $-n$ with multiplicity $M(d, n)$. 

Martin and Wagner show they are integral when $d \leq 6$ (and $n \leq \left(\frac{d^2}{2}\right)$) and conjecture they are always integral. We show that they are integral for $n \leq 8$ for any value of $d$ (such that $n \leq \left(\frac{d^2}{2}\right)$).
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We show that they are integral for $n \leq 8$ for any value of $d$ (such that $n \leq \binom{d}{2}$).
Future Work

- Find the spectrum of $SR(d, n)$ for more cases.

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- Prove (or disprove) that the partial permutohedra are always integral.
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- Prove (or disprove) that the partial permutohedra are always integral.
- Find the independence number of $SR(d, n)$, which is the number of mutually nonattacking rooks which can be placed on a $(d - 1)$-dimensional simplicial chessboard with $n + 1$ tiles on each side (known only for $d = 3$).