On circulant graphs isomorphic to Cayley graphs of more than one abelian group

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Joint work with Joy Morris of the University of Lethbridge
Definition

Let $G$ be a group and $S \subset G$. 

A Cayley digraph of a cyclic group of order $n$ is circulant digraph of order $n$.
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Figure: The Cayley graph $\text{Cay}(\mathbb{Z}_{10}, \{1, 3, 7, 9\})$
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Let $G$ be a group and $S \subset G$. Define a Cayley digraph of $G$, denoted $\text{Cay}(G, S)$, to be the digraph with $V(\text{Cay}(G, S)) = G$ and $E(\text{Cay}(G, S)) = \{(g, gs) : g \in G, s \in S\}$. 
Let $G$ be a group and $S \subset G$. Define a **Cayley digraph of $G$**, denoted $\text{Cay}(G, S)$, to be the digraph with $V(\text{Cay}(G, S)) = G$ and $E(\text{Cay}(G, S)) = \{(g, gs) : g \in G, s \in S\}$. We call $S$ the **connection set of** $\text{Cay}(G, S)$.
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\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{cayley_graph.png}
\caption{The Cayley graph $\text{Cay}(\mathbb{Z}_{10}, \{1, 3, 7, 9\})$}
\end{figure}
Definition

For a group $G$, the \textit{left regular representation of $G$},

$$G_L = \{ x \rightarrow gx : g \in G \}.$$
For a group $G$, the \textit{left regular representation} of $G$, denoted $G_L$, is the subgroup of $S_G$ given by the left translations of $G$. More specifically, $G_L = \{ x \rightarrow gx : g \in G \}$. We denote the map $x \rightarrow gx$ by $g_L$. It is straightforward to verify that $G_L$ is a group and that $G_L \cong G$. It is easy to show $G_L \leq \text{Aut}(\text{Cay}(G, S))$ for every $S \subset G$. 

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$\{(u, v)(u', v') : u \in V(\Gamma_1) \text{ and } vv' \in E(\Gamma_2)\} \cup \{(u, v)(u', v') : uu' \in E(\Gamma_1) \text{ and } v, v' \in V(\Gamma_2)\}.$

Intuitively, $\Gamma_1 \wr \Gamma_2$ is constructed as follows. First, we have $|V(\Gamma_1)|$ copies of the digraph $\Gamma_2$, with these $|V(\Gamma_1)|$ copies indexed by elements of $V(\Gamma_1)$. Next, between corresponding copies of $\Gamma_2$ we place every possible directed from one copy to another if in $\Gamma_1$ there is an edge between the indexing labels of the copies of $\Gamma_2$, and no edges otherwise.
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To find the wreath product of any two digraphs $\Gamma_1$ and $\Gamma_2$:

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\[ \widehat{\Gamma_1 \wr \Gamma_2} = (0, a) \cdot (0, b) \cdot (0, c) \cdot (0, d) \cdot (1, a) \cdot (1, b) \cdot (1, c) \cdot (1, d) \]
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\begin{align*}
\Gamma_1 & \quad \Gamma_1 \wr \Gamma_2 \\
0 & \quad (0, a) \\
1 & \quad (0, b) \\
a & \quad (0, c) \\
d & \quad (0, d) \\
b & \quad (1, a) \\
c & \quad (1, b) \\
d & \quad (1, c) \\
& \quad (1, d)
\end{align*}
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To find the wreath product of any two digraphs $\Gamma_1$ and $\Gamma_2$:

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\[
\begin{align*}
\Gamma_1 \wr \Gamma_2 & \approx \begin{pmatrix}
(0, a) & (1, a) \\
(0, b) & (1, b) \\
(0, c) & (1, c) \\
(0, d) & (1, d)
\end{pmatrix}
\end{align*}
\]
Let us consider the graph $C_8 \wr \overline{K}_2$. 
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![Diagram](attachment:image_url)

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Theorem (Morris, 1998)

Let \( \Gamma = \text{Cay}(G, S) \) be a Cayley digraph on an abelian group \( G \) of order \( p^n \), where \( p \) is an odd prime. Then the following are equivalent:

1. The digraph \( \Gamma \) is isomorphic to a Cayley digraph on both \( \mathbb{Z}_{p^n} \) and \( H \), where \( H \) is an abelian group with \( |H| = p^n \), say
   \[ H = \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}} \times \ldots \times \mathbb{Z}_{p^{k_m}}, \]
   where \( k_1 + \ldots + k_m = n \).

2. There exist a chain of subgroups \( G_1 \leq \ldots \leq G_{m-1} \) in \( G \) such that
   1. \( G_1, G_2/G_1, \ldots, G/G_{m-1} \) are cyclic groups;
   2. \( G_1 \times G_2/G_1 \times \ldots \times G/G_{m-1} \triangleleft_{po} H; \)
   3. For all \( s \in S \setminus G_i \), we have \( sG_i \subseteq S \), for \( i = 1, \ldots, m-1 \). (That is, \( S \setminus G_i \) is a union of cosets of \( G_i \).)
3. There exist Cayley digraphs $U_1, \ldots, U_m$ on cyclic $p$-groups $H_1, \ldots, H_m$ such that $H_1 \times \ldots \times H_m \prec_{po} H$ and $\Gamma \cong U_m \wr \ldots \wr U_1$.

These in turn imply:

4. $\Gamma$ is isomorphic to Cayley digraphs on every abelian group of order $p^n$ that is greater than $H$ in the partial order.
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These in turn imply:

4. $\Gamma$ is isomorphic to Cayley digraphs on every abelian group of order $p^n$ that is greater than $H$ in the partial order.

This generalized work of Joseph (1995) $n = 2$. The case where $p = 2$ was recently proven by Kovács and Servetius (2012).
The idea behind this paper is to reduce the case for general $n$ in the above theorem to the prime-power case.

By Sabidussi, we would have transitive groups $G_1$ and $G_2$ of order $n$ in $\text{Aut}(\Gamma)$, $G_1$ cyclic and $G_2$ abelian. We would want the Sylow $p$-subgroup of $G_1$ to "interact" with the Sylow $p$-subgroup of $G_2$, but the $p'$-subgroup of $G_1$ to commute with the Sylow $p$-subgroup of $G_2$ and the $p'$-subgroup of $G_2$ to commute with the Sylow $p$-subgroup of $G_1$.

Another way of phrasing this is that we would like $\langle G_1, G_2 \rangle$ to be nilpotent!
The idea behind this paper is to reduce the case for general $n$ in the above theorem to the prime-power case. We need the characterization of Cayley digraphs by Sabidussi:

A digraph $\Gamma$ is isomorphic to a Cayley digraph of a group $G$ of order $n$ if and only if $\text{Aut}(\Gamma)$ contains a transitive subgroup isomorphic to $G$.

What would it mean in this context for the problem to reduce to the prime-power case? By Sabidussi, we would have transitive groups $G_1$ and $G_2$ of order in $\text{Aut}(\Gamma)$, $G_1$ cyclic and $G_2$ abelian. We would want the Sylow $p$-subgroup of $G_1$ to "interact" with the Sylow $p$-subgroup of $G_2$, but the $p'$-subgroup of $G_1$ to commute with the Sylow $p$-subgroup of $G_2$ and the $p'$-subgroup of $G_2$ to commute with the Sylow $p$-subgroup of $G_1$.

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By Sabidussi, we would have transitive groups $G_1$ and $G_2$ of order in $\sqrt{n}$ in $\text{Aut}(\Gamma)$, $G_1$ cyclic and $G_2$ abelian. We would want the Sylow $p$-subgroup of $G_1$ to "interact" with the Sylow $p$-subgroup of $G_2$, but the $p'$-subgroup of $G_1$ to commute with the Sylow $p$-subgroup of $G_2$ and the $p'$-subgroup of $G_2$ to commute with the Sylow $p$-subgroup of $G_1$.

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Theorem (Muzychuk, 1998)

Let $G_1$ and $G_2$ be transitive cyclic subgroups of $S_n$ of order $n$. Then there exists $\delta \in \langle G_1, G_2 \rangle$ such that $\langle G_1, \delta^{-1}G_2\delta \rangle$ is solvable and normally $m$-step imprimitive.

Theorem

Let $G_1$ and $G_2$ be transitive abelian subgroups of $S_n$ of order $n$ with $G_1$ cyclic. Then there exists $\delta \in \langle G_1, G_2 \rangle$ such that $\langle G_1, \delta^{-1}G_2\delta \rangle$ is solvable and normally $m$-step imprimitive.
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Let $k = p_1 \ldots p_r$ be such that $\gcd(k, \varphi(k)) = 1$ where each $p_i$ is prime, and $n = p_1^{a_1} \ldots p_r^{a_r}$. Let $\Gamma = \text{Cay}(G, S)$ be a Cayley digraph on an abelian group $G$ of order $n$. Then the following are equivalent:

1. The digraph $\Gamma$ is isomorphic to a Cayley digraph on both $\mathbb{Z}_n$ and $H$, where $H$ is an abelian group with $|H| = n$, say $H = \prod_{i=1}^{r} \prod_{j=1}^{m_i} \mathbb{Z}_{p_i^{k_{i,j}}}$, where $\sum_{j=1}^{m_i} k_{i,j} = a_i$.

2. Let $P_i$ be a Sylow $p_i$-subgroup of $G$. There exist a chain of subgroups $P_{i,1} \leq \ldots \leq P_{i,m_i-1}$ in $P_i$ such that
   - $P_{i,1}, P_{i,2}/P_{i,1}, \ldots, P_{i}/P_{i,m_i-1}$ are cyclic groups of prime-power order;
   - $P_{i,1} \times P_{i,2}/P_{i,1} \times \ldots \times P_{i}/P_{i,m_i-1} \vartriangleleft_p H_i$, where $H_i$ is a Sylow $p_i$-subgroup of $H$;
   - For all $s \in S \setminus (P_{i,j} \times G'_{i})$, we have $sP_{i,j} \subseteq S$, for $j = 1, \ldots, m_i - 1$, where $G'_{i}$ is a $p'_i$-subgroup of $G$. (That is, $S \setminus (P_{i,j} \times G'_{i})$ is a union of cosets of $P_{i,j}$.)
Theorem

3. There exist Cayley digraphs $U_{i,1}, \ldots, U_{i,m_i}$ on cyclic $p_i$-groups $K_{i,1}, \ldots, K_{i,m_i}$ such that $K_{i,1} \times \ldots \times K_{i,m_i} \preceq_{po} H_i$, $\Gamma_i \cong U_{i,m_i} \wr \ldots \wr U_{i,1}$, and $\Gamma$ is of product type $\Gamma_1, \ldots, \Gamma_r$.

These in turn imply:

4. $\Gamma$ is isomorphic to Cayley digraphs on every abelian group of order $n$ that is greater than $H$ in the partial order.
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Definition

Let $\Gamma_1, \ldots, \Gamma_k$ be digraphs. We say that $\Gamma$ is of product type $\Gamma_1, \ldots, \Gamma_k$ if $\text{Aut}(\Gamma_1) \times \ldots \times \text{Aut}(\Gamma_k) \leq \text{Aut}(\Gamma)$. 
The “input” into the proof of the previous result is that $\langle (\mathbb{Z}/n\mathbb{Z})^L, H \rangle$ is nilpotent, and so does not depend in general on the value of $n$. To reduce the general case to prime-powers one would need to generalize Muzychuk’s solution of the isomorphism problem for circulants. If one wished to reduce the general case to prime-powers using group theoretic techniques, then the generalization of a theorem of Muzychuk presented here would be a natural first step.

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